First-order transition with power-law singularity in models with absorbing states

Adam Lipowski*
Department of Physics, Adam Mickiewicz University, ulica Umultowska 85, 61-614 Poznań, Poland
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We study one- and two-dimensional models which undergo a transition between active and absorbing phases. The transition point in these models is of a novel type: jump of the order parameter coincides with its power-law singularity. Some arguments supported by Monte Carlo simulations prompted us to predict the exact location of the transition point. Both models possess gaugelike symmetry.

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I. INTRODUCTION

Recently, nonequilibrium phase transitions have been intensively studied in a variety of models [1]. In addition to some potential applications, the motivation to study these transitions comes from the belief that they can be categorized into universality classes similarly to equilibrium phase transitions. In this context, models which exhibit transitions between active and absorbing phases [2] are of particular interest. There already exists substantial numerical evidence that phase transitions in such models indeed can be classified into some universality classes. For example, it is believed that models with unique absorbing states should belong to the so-called directed-percolation (DP) universality class [3]. Moreover, models with double (symmetric) absorbing states or with some conservation law in their dynamics belong to another universality class [4].

Similarly to equilibrium systems, nonequilibrium continuous phase transitions are not the only possibility—some models are known to undergo discontinuous transitions [5–7]. Although such transitions are not classified into universality classes, they might be more relevant since discontinuous transitions are, at present, the only type of transitions which can be observed experimentally. On the contrary, the experimental realization of continuous phase transitions still remains an open problem [8].

One reason for a relatively good understanding of equilibrium phase transitions is a wealth of exactly solvable models in this field [9]. In this respect, the situation is much worse for nonequilibrium phase transitions. None of the models with absorbing states and with continuous or discontinuous transitions was solved exactly and all results concerning the critical exponents or the location of a transition point are only numerical.

In the present paper we study certain models with infinitely many absorbing states. At a certain value of a control parameter $r=r_c$, these models undergo a transition between active and absorbing phases. But the interesting point is a novel type of this transition: it seems to combine some features of both discontinuous and continuous transitions. Namely, at $r=r_c$ an order parameter jumps discontinuously to zero, but in addition to that the order parameter has a power-law singularity upon approaching the transition point from the active phase. Moreover, some elementary arguments, supported by Monte Carlo simulations, prompted us to predict the exact location of the transition point in both models, namely, $r_c=0$. Both models have a gaugelike symmetry, which might be responsible for the unusual behavior of these models.

II. SQUARE LATTICE

Our first model is a certain modification of a model introduced in the context of modeling biological evolution [10,11]. It is defined on a two-dimensional ($d=2$) square lattice where for each bond between the nearest-neighboring sites i and j we introduce bond variables $w_{i,j}\in(-0.5,0.5)$. Introducing the parameter r, we call the site i active when $\Pi_j w_{i,j}<r$, where j runs over all nearest neighbors of i. Otherwise, the site is called nonactive. The model is driven by random sequential dynamics, and when the active site i is selected, we assign anew, with uniform probability, four bond variables $w_{i,j}$, where j is one of the nearest neighbors of i. Nonactive sites are not updated, but updating a certain (active) site might change the status of its neighbors.

An important quantity characterizing this model is the steady-state density of active sites ρ. How does ρ change with the control parameter r? Of course, for $r\geqslant(0.5)^4=0.0625$, all sites are active ($\rho=1$) for any distribution of bond variables $w_{i,j}$. It is natural to expect that for $r<0.0625$ and not too small there will be a certain fraction of active sites ($\rho>0$) and this fraction will decrease when r decreases. Since the dynamical rules imply that the model has absorbing states with all sites nonactive ($\rho=0$), one can expect that at a certain r the model undergoes a transition between the active and absorbing phases. In general, one expects that this transition might be either continuous and presumably of (2+1)DP universality class [12] or discontinuous.

The existence of a transition is confirmed in Fig. 1, which shows the density ρ as a function of r obtained using Monte Carlo simulations. The simulations were performed for the linear system size $L=300$ and we checked that the presented results are, within small statistical error, size-independent. After relaxing the random initial configuration for $t_{\rm rel}=10^4$, we made measurements during runs of $t=10^5$ (the unit of time is defined as a single on average update/lattice site). From this figure one can also see that the transition point r_c is located very close to $r=0$ and in the following we are going to show that it is very likely that in this model $r_c=0$ (exactly).

*Email address: lipowski@main.amu.edu.pl

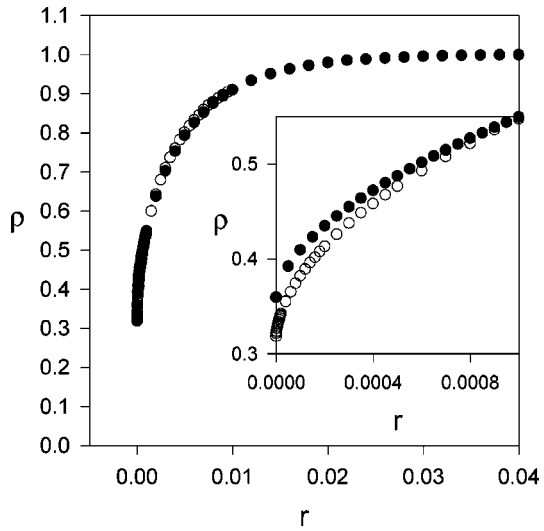

FIG. 1. The steady-state density of active sites ρ as a function of r for the $d=1$ (○) and $d=2$ (●) models. The simulations were made for the system size $L=300$ ($d=2$) and up to $L=3\times10^5$ ($d=1$). For $r<0$, the system quickly reaches an absorbing state (see Fig. 2). The error bars are smaller than the plotted symbols. The inset shows our data in the vicinity of $r=0$. For $r>10^{-3}$, the density ρ for both models is almost the same.

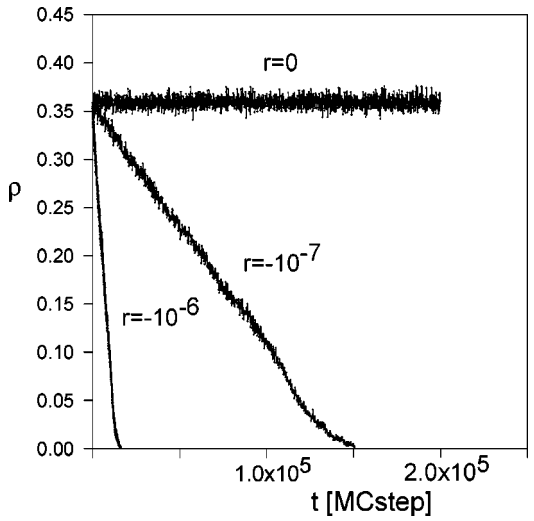

FIG. 2. The time evolution of the density of active sites $\rho(t)$ ($d=2$ and $L=100$). For $r=0$ the system remains in the active phase, but as soon as r becomes negative it evolves toward an absorbing state. For $r=0$ and $L=300$, the simulations give, within small statistical errors, the same results.

First, we show that for $r<0$, the model is in the absorbing phase. The argument for that is elementary and based on the following observation: for $r<0$, there exists a finite probability that after updating, a given site will become nonactive forever. Indeed, when one of the newly selected bonds satisfies the condition

$$|w_{i,j}|<-r/(0.5)^3, \tag{1}$$

then the sites i and j become permanently nonactive (i.e., regardless of the other bonds attached to these sites, they will always remain nonactive). For $r<0$, there is a finite probability to satisfy Eq. (1), and the above mechanism leads to a rapid decrease of active sites and hence the system reaches an absorbing state. The above mechanism is not effective for $r\geqslant0$ since there is no value which would ensure permanent nonactivity of a certain site.

To confirm that for $r<0$ the system is in the absorbing phase, we present in Fig. 2 the time evolution of ρ for $r=-10^{-6}$ and -10^{-7}. Although these values are very close to $r=0$, one can clearly see that the system evolves toward the absorbing state. (For r smaller than these values, the approach to the absorbing state would be even faster.) As we have already mentioned, for $r\geqslant0$ the mechanism which generates permanently nonactive sites is not effective. Most likely, this has important consequences: as shown in Fig. 2, even for $r=0$ the system does not evolve toward the absorbing state but remains in the active phase.

These results indicate that at $r=0$ the model undergoes a first-order transition between active and absorbing phases, characterized by a discontinuity of the order parameter ρ. However, the most interesting feature of the model is the fact that upon approaching the first-order transition point $r=0$, the order parameter exhibits a power-law singularity. Such singularities usually signal a continuous transition. This singularity, which is already visible in the inset of Fig. 1, is also presented in the logarithmic plot in Fig. 3. The parameter $\rho_0=0.359$ (i.e., the density of active sites for $r=0$) in Fig. 3 was obtained from the least-square analysis of small-r ($r\leqslant10^{-3}$) data shown in Fig. 1 using the formula

$$\rho(r)=\rho_0+Ar^{\beta}, \tag{2}$$

where we assumed that the critical point is located at $r=0$ [14]. From the slope of the data in Fig. 3, we estimate $\beta=0.58(1)$, which might suggest that the exponent β for that model is the same as in the (2+1)DP [13,14]. However, a characteristic feature of models of the DP universality class is that at the transition point the model falls into an absorbing

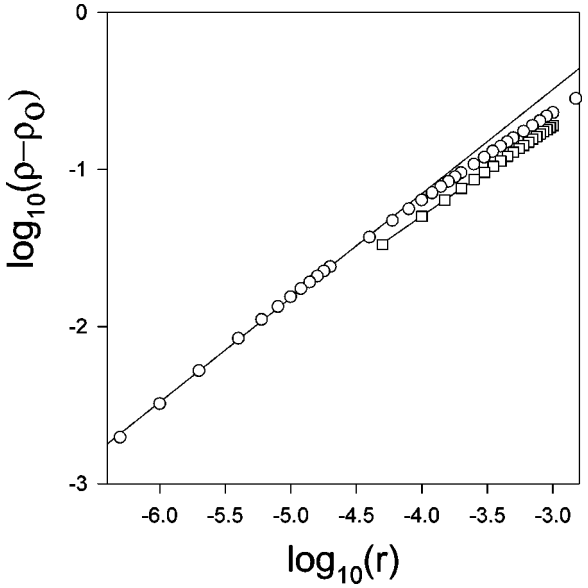

FIG. 3. The plot of $\log_{10}(\rho-\rho_0)$ as a function of $\log_{10}(r)$ with $p_0=0.359$ ($d=2$, □) and $p_0=0.314\,827$ ($d=1$, ○). The lines have slopes corresponding to $\beta=0.58$ ($d=2$) and $\beta=0.66$ ($d=1$).

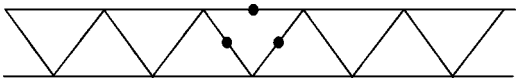

FIG. 4. The triangular ladder. When three bonds (●) around a certain triangle are inverted, activity of the system remains unchanged.

state. Our model at the transition point ($r=0$) is not in the absorbing phase (see Fig. 2), but it enters the absorbing phase as soon as r becomes negative. In addition, scaling behavior of our numerical data persists on a relatively small interval of r, and asymptotically a different behavior might set in. Further arguments against DP criticality of this model are presented in the next section.

Let us notice that the above model is characterized by very large gaugelike symmetry. Indeed, inverting ($\pm$) four bond variables around any elementary square does not change the activity of the sites. The gauge symmetry was examined for many equilibrium lattice models [15]. However, the models examined so far with absorbing states do not possess this kind of symmetry. It would be interesting to check whether the unusual properties of this model are related with this symmetry. In the following, we examine a one-dimensional model which possesses a similar symmetry.

III. TRIANGULAR LADDER

Let us examine a model defined on a one-dimensional ($d=1$) ladderlike lattice, where each site also has four neighbors (see Fig. 4). When defined with the same dynamical rules as the model examined in the preceding section, this $d=1$ model also has an analogous gauge symmetry (see Fig. 4).

We examined the properties of this model using the same Monte Carlo method. Results of our simulations for the steady-state density ρ are shown in Fig. 1 and Fig. 3. As is usually the case in models with absorbing states, Monte Carlo simulations of the $d=1$ version are much more accurate. For example, close to and at the transition point $r=0$ we simulated the system of the size $L=3\times10^5$, and the simulation time was typically $t=10^6$. As a result, we were able to probe a much closer vicinity of the transition point.

Our results indicate that the behavior of $d=1$ and $d=2$ versions of this model is very similar. Both models exhibit qualitatively the same transition at $r=0$. In the $d=1$ case, our estimations of the critical parameters are $\rho_0=0.314\,827$ and $\beta=0.66(3)$. A relatively good scaling behavior in this case is confirmed over two decades (see Fig. 3). The obtained value of the exponent β clearly excludes the DP value (in the case of one-dimensional DP $\beta=0.276\,486$ [16]).

To get further insight into the nature of the transition point, we examined the size dependence of the relaxation time τ. We measured the time needed for the system starting from the random initial configuration to reach the steady state. Typically, at the continuous transition τ diverges as L^z, where z is a positive exponent. For the one- and two-dimensional DP, $z=1.58$ and 1.76, respectively. At the discontinuous transition one expects that τ remains finite in the thermodynamic limit (i.e., $z=0$). For $r=0$ and $d=1$, results of our measurements, shown in Fig. 5, are, in our opinion, inconclusive. They may suggest a power-law divergence

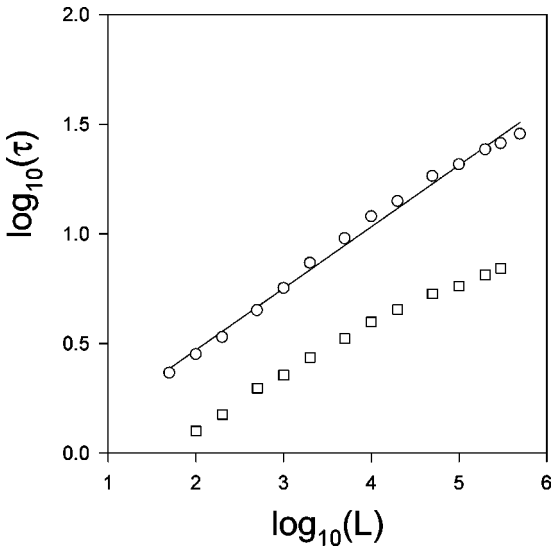

FIG. 5. The size dependence of the relaxation time τ for the $d=1$ model and $r=0$ (○) and 10^{-3} (□). The straight line has a slope 0.3.

with a small exponent $z(\sim0.2)$, but positive curvature of our data might asymptotically lead to $z=0$. On the other hand, even if $z=0$, it is not certain whether τ remains finite or diverges, but is slower than a power of L. For $r>0$ (i.e., off-criticality), the numerical data are similar to $r=0$, but in general one expects that τ remains finite in the thermodynamic limit.

IV. SUMMARY

In the present paper we studied two models which exhibit remarkably similar and unusual behavior. These models have a transition point which, although mainly of discontinuous nature (jump of the order parameter and $z=0$), has a certain feature of continuous transitions (power-law singularity of the order parameter).

The main weakness of our paper is the lack of any theoretical argument which would explain the behavior of these models. Both models possess certain gaugelike symmetry. The role of such symmetry in models with absorbing states was not yet explored and it is possible that the behavior of these models might be related with this symmetry.

Are there any indications that such transitions might take place in real systems? In our opinion, one of the possible applications might be related with phase transitions in nuclear physics. Indeed, there are some indications that multifragmentation of heavy nuclei resembles a phase transition which has both first- and second-order features [17]. Such systems have been already modeled using Ising-like models. However, such an approach implicitly assumes a thermalization of the system, which is not obvious in these multifragmentation processes. Models with absorbing states might provide an alternative description of such processes.

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